Eigenvalues and Eigenvectors Linear Algebra X. Du

- Suppose you have a square matrix *A*. Let *A* represent a linear transformation that is multiplied by a vector $\vec{v}$ (i.e. let *A* be a transformation matrix upon another vector $\vec{v}$).
  - If $A\vec{v}$ is parallel to $\vec{v}$ (i.e. the transformed vector is parallel to the original vector), then $\vec{v}$ is an **eigenvector** of *A*.
    - By convention, $\vec{v} \neq \vec{0}$.
  - Consequently, the constant factor by which the magnitude of the vector has changed is the **eigenvalue** associated with $\vec{v}$ and *A*.
  - i.e. $A\vec{v} = \lambda\vec{v}$, where $\vec{v}$ is the eigenvector and λ is the associated eigenvalue.
- **Characteristic polynomial** $= \det(A - \lambda I)$
  - Where does this come from?
    - $A\vec{v} = \lambda\vec{v} \Rightarrow A\vec{v} - \lambda\vec{v} = \vec{0} \Rightarrow (A - \lambda I)\vec{v} = \vec{0}$, where *I* is the identity matrix.
    - $(A - \lambda I)\vec{v} = \vec{0}$ iff $\det(A - \lambda I) = 0$. (Assume $\vec{v} \neq \vec{0}$)
      - $\det(A - \lambda I)$ is the **characteristic polynomial** of *A*.
  - For a two-by-two matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.
    - $\det(A - \lambda I) = (a - \lambda)(d - \lambda) - bc = \lambda^2 - (a + d)\lambda + ab - bc = 0$
    - $\lambda^2 - (a + d)\lambda + (ab - bc) = 0$
      - Characteristic equation can be written as $\lambda^2 - T\lambda + D = 0$.
- How to find eigenvalues and eigenvectors:
  - 1. Find the characteristic polynomial of *A* (i.e. find $\det(A - \lambda I)$).
  - 2. Solve $\det(A - \lambda I) = 0$ to obtain a set of eigenvalues.
  - 3. For each eigenvalue, find an associated eigenvector by substituting back into the equation $(A - \lambda I)\vec{v} = \vec{0}$ and solving the system of equations.
    - The system of equations should be redundant (i.e. each individual equation in the system should be linearly dependent on all the others).
    - Note: Any eigenvector will do, as every eigenvector associated with a specific eigenvalue will be linearly dependent. But for practical purposes, choose the most reduced eigenvector.
- The sum of the eigenvalues is the trace, the product of eigenvalues is the determinant
- **Defective matrix**: an $n \times n$ matrix *A* that does not have *n* independent eigenvectors
- For a triangular matrix, the eigenvalues are the diagonal entries.
- If λ and $\vec{v}$ are corresponding eigenvalues and eigenvectors for *A*, then λ^k and $\vec{v}$ are corresponding eigenvalues and eigenvectors for A^k.
- *A* is invertible iff 0 is not an eigenvalue of *A*.
- Applications
  - Solving systems of differential equations
  - Transforming images (e.g. scaling, rotating, etc.)
  - Vibration analysis
  - Computational chemistry: Schrödinger equation, molecular orbital theory